A GENERAL DIVERGENCE MEASURE FOR MONOTONIC FUNCTIONS AND APPLICATIONS IN INFORMATION THEORY

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Abstract. A general divergence measure for monotonic functions is introduced. Its connections with the $f-$divergence for convex functions are explored. The main properties are pointed out.

1. Introduction

Let $(X,\mathcal{A})$ be a measurable space satisfying $|\mathcal{A}|>2$ and μ be a $\sigma-$finite measure on $(X,\mathcal{A})$. Let $\mathcal{P}$ be the set of all probability measures on $(X,\mathcal{A})$ which are absolutely continuous with respect to μ. For $P,Q\in\mathcal{P}$, let $p=\frac{dP}{d\mu}$ and $q=\frac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ.

Two probability measures $P,Q\in\mathcal{P}$ are said to be *orthogonal* and we denote this by $Q\perp P$ if

$$P(\{q=0\})=Q(\{p=0\})=1.$$

Let $f:[0,\infty)\to(-\infty,\infty]$ be a convex function that is continuous at 0, i.e., $f(0)=\lim_{u\downarrow 0}f(u)$.

In 1963, I. Csiszár [2] introduced the concept of $f-$divergence as follows.

Definition 1. *Let $P,Q\in\mathcal{P}$. Then*

$$I_f(Q,P)=\int_X p(x)\,f\left[\frac{q(x)}{p(x)}\right]d\mu(x), \tag{1.1}$$

is called the $f-$divergence of the probability distributions Q and P.

We now give some examples of $f-$divergences that are well-known and often used in the literature (see also [3]).

1.1. The Class of $\chi^\alpha-$Divergences.

The $f-$divergences of this class, which is generated by the function χ^α, $\alpha\in[1,\infty)$, defined by

$$\chi^\alpha(u)=|u-1|^\alpha,\quad u\in[0,\infty)$$

have the form

$$I_f(Q,P)=\int_X p\left|\frac{q}{p}-1\right|^\alpha d\mu=\int_X p^{1-\alpha}|q-p|^\alpha d\mu. \tag{1.2}$$

From this class only the parameter $\alpha=1$ provides a distance in the topological sense, namely the *total variation distance* $V(Q,P)=\int_X|q-p|\,d\mu$. The most prominent special case of this class is, however, *Karl Pearson's χ^2-divergence.*

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1.2. **Dichotomy Class.** From this class, generated by the function $f_\alpha : [0,\infty) \to \mathbb{R}$

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \frac{1}{\alpha(1-\alpha)}\left[\alpha u + 1 - \alpha - u^\alpha\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ 1 - u + u \ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ $\left(f_{\frac{1}{2}}(u) = 2\left(\sqrt{u}-1\right)^2\right)$ provides a distance, namely, the *Hellinger distance*

$$H(Q,P) = \left[\int_X \left(\sqrt{q} - \sqrt{p}\right)^2 d\mu\right]^{\frac{1}{2}}.$$

Another important divergence is the *Kullback-Leibler divergence* obtained for $\alpha = 1$,

$$KL(Q,P) = \int_X q \ln\left(\frac{q}{p}\right) d\mu.$$

1.3. **Matsushita's Divergences.** The elements of this class, which is generated by the function φ_α, $\alpha \in (0,1]$ given by

$$\varphi_\alpha(u) := \left|1 - u^\alpha\right|^{\frac{1}{\alpha}}, \quad u \in [0,\infty),$$

are prototypes of metric divergences, providing the distances $\left[I_{\varphi_\alpha}(Q,P)\right]^\alpha$.

1.4. **Puri-Vineze Divergences.** This class is generated by the functions Φ_α, $\alpha \in [1,\infty)$ given by

$$\Phi_\alpha(u) := \frac{|1-u|^\alpha}{(u+1)^{\alpha-1}}, \quad u \in [0,\infty).$$

It has been shown in [4] that, this class provides the distances $\left[I_{\Phi_\alpha}(Q,P)\right]^{\frac{1}{\alpha}}$.

1.5. **Divergences of Arimoto-type.** This class is generated by the functions

$$\Psi_\alpha(u) := \begin{cases} \frac{\alpha}{\alpha-1}\left[(1+u^\alpha)^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}-1}(1+u)\right] & \text{for } \alpha \in (0,\infty)\setminus\{1\}; \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for } \alpha = 1; \\ \frac{1}{2}|1-u| & \text{for } \alpha = \infty. \end{cases}$$

It has been shown in [5] that, this class provides the distances $\left[I_{\Psi_\alpha}(Q,P)\right]^{\min\left(\alpha,\frac{1}{\alpha}\right)}$ for $\alpha \in (0,\infty)$ and $\frac{1}{2}V(Q,P)$ for $\alpha = \infty$.

2. Some Classes of Normalised Functions

We denote by $\mathcal{M}^{\ddagger}([0,\infty))$ the class of *monotonic nondecreasing functions* defined on $[0,\infty)$ and by $\mathcal{M}s([0,\infty))$ the class of *measurable functions* on $[0,\infty)$. We also consider $\mathcal{L}e_1([0,\infty))$ the class of measurable functions $g : [0,\infty) \to \mathbb{R}$ with the property that

$$g(t) \le g(1) \le g(s) \quad \text{for } 0 \le t \le 1 \le s < \infty. \tag{2.1}$$

It is obvious that

$$\mathcal{M}^{\ddagger}([0,\infty)) \subsetneqq \mathcal{L}e_1([0,\infty)), \tag{2.2}$$

and the inclusion (2.2) is strict.

We say that a function $f:[0,\infty)\to\mathbb{R}$ is *normalised* if $f(1)=0$. We denote by $\mathcal{M}s_0([0,\infty))$ the class of all normalised measurable functions defined on $[0,\infty)$. We also need the following classes of functions

$$\mathcal{C}o([0,\infty)) := \{f\in\mathcal{M}s_0([0,\infty))\,|\,f \text{ is continuous convex on } [0,\infty)\};$$

$$\mathcal{D}_0([0,\infty)) := \left\{f\in\mathcal{M}s_0([0,\infty))\,|\,f(t)=(t-1)\,g(t),\ \forall t\in[0,\infty),\ g\in\mathcal{M}^{\ddagger}([0,\infty))\right\};$$

and

$$\mathcal{O}_0([0,\infty)) := \{f\in\mathcal{M}s_0([0,\infty))\,|\,f(t)=(t-1)\,g(t),\ \forall t\in[0,\infty),\ g\in\mathcal{L}e_1([0,\infty))\}.$$

From the definition of $\mathcal{D}_0([0,\infty))$ and $\mathcal{O}_0([0,\infty))$ and taking into account that the strict inclusion (2.2) holds, we deduce that

$$\mathcal{D}_0([0,\infty)) \subsetneqq \mathcal{O}_0([0,\infty)), \tag{2.3}$$

and the inclusion is strict.

For the other two classes, we may state the following result.

Lemma 1. *We have the strict inclusion*

$$\mathcal{C}o([0,\infty)) \subsetneqq \mathcal{D}_0([0,\infty)). \tag{2.4}$$

Proof. We will show that any continuous convex function $f:[0,\infty)\to\mathbb{R}$ that is normalised may be represented as:

$$f(t)=(t-1)\,g(t)\ \text{ for any }\ t\in[0,\infty), \tag{2.5}$$

where $g\in\mathcal{M}^{\ddagger}([0,\infty))$.

Now, let $f\in\mathcal{C}o([0,\infty))$. For $\lambda\in[D_-f(1),D_+f(1)]$, define

$$g_\lambda(t) := \begin{cases} \dfrac{f(t)}{t-1} & \text{if }\ t\in[0,1)\cup(1,\infty),\\[2ex] \lambda & \text{if }\ t=1.\end{cases}$$

We use the following well known result [1, p. 111]:

If Ψ is convex on (a,b) and $a<s<t<u<b$, then

$$\Psi(s,t)\le\Psi(s,u)\le\Psi(t,u), \tag{2.6}$$

where

$$\Psi(s,t)=\frac{\Psi(t)-\Psi(s)}{t-s}.$$

If Ψ is strictly convex on (a,b), equality will not occur in (2.6).

If we apply the above result for $0<s<t<1$, then we can state

$$\frac{f(s)}{s-1}\le\frac{f(t)}{t-1}.$$

Taking the limit over $t\to1$, $t<1$, we deduce

$$\frac{f(s)}{s-1}\le D_-f(1)$$

showing that for $0<t<1$, we have $g_\lambda(t)\le\lambda$.

Similarly, we may prove that for $1<t<\infty$, $g_\lambda(t)\ge\lambda$. If we use the same result for $0<t_1<t_2<1$, then we may write

$$\frac{f(t_1)}{t_1-1}\le\frac{f(t_2)}{t_2-1},$$

which gives $g_\lambda(t_1) \le g_\lambda(t_2)$ for $0 < t_1 < t_2 < 1$.

In a similar fashion we can prove that for $1 < t_1 < t_2 < \infty$, $g_\lambda(t_1) \le g_\lambda(t_2)$, and thus we may conclude that the function g_λ is monotonic non-decreasing on the whole interval $[0,\infty)$.

If we consider now the function $f(t) = (t-1)e^{\eta t}$, $t \in [0,\infty)$, we observe that $f'(t) = (\eta t - 3)e^{\eta t}$, $f''(t) = 8e^{\eta t}(2t-1)$ which shows that f is not convex on $[0,\infty)$. Obviously, $f \in \mathcal{D}_0([0,\infty))$, and thus the inclusion (2.4) is indeed strict. ∎

Remark 1. *If $f \in \mathcal{D}_0([0,\infty))$ and $g_1, g_2 \in \mathcal{M}^{\ddagger}([0,\infty))$ are two functions with*

$$f(t) = (t-1)\,g_1(t), \quad f(t) = (t-1)\,g_2(t)$$

for each $t \in [0,\infty)$, then we get

$$(t-1)[g_1(t) - g_2(t)] = 0$$

for any $t \in [0,\infty)$ showing that $g_1(t) = g_2(t)$ for each $t \in [0,1) \cup (1,\infty)$. They may have different values in $t = 1$.

3. Some Fundamental Properties of $f-$Divergence for $f \in \mathcal{C}o([0,\infty))$

For $f \in \mathcal{C}o([0,\infty))$ we obtain the $*-$*conjugate* function of f by

$$f^*(u) = uf\left(\frac{1}{u}\right), \quad u \in (0,\infty).$$

It is also known that if $f \in \mathcal{C}o([0,\infty))$, then $f^* \in \mathcal{C}o([0,\infty))$.

The following two theorems contain the most basic properties of $f-$divergences. For their proof we refer the reader to Chapter 1 of [6] (see also [3]).

Theorem 1 (Uniqueness and Symmetry Theorem)**.** *Let f, f_1 be continuous convex on $[0,\infty)$.*

(i) *We have*

$$I_{f_1}(Q,P) = I_f(Q,P),$$

for any $P, Q \in \mathcal{P}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$f_1(u) = f(u) + c(u-1),$$

for any $u \in [0,\infty)$;

(ii) *We have*

$$I_{f^*}(Q,P) = I_f(Q,P),$$

for any $P, Q \in \mathcal{P}$ if and only if there exists a constant $d \in \mathbb{R}$ such that

$$f^*(u) = f(u) + d(c-1),$$

for any $u \in [0,\infty)$.

Theorem 2 (Range of Values Theorem)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous convex function on $[0,\infty)$.*

For any $P, Q \in \mathcal{P}$, we have the double inequality

$$f(1) \le I_f(Q,P) \le f(0) + f^*(0). \tag{3.1}$$

(i) *If $P = Q$, then the equality holds in the first part of (3.1).*

If f is strictly convex at 1, then the equality holds in the first part of (3.1) if and only if $P = Q$;

(ii) *If $Q \perp P$, then the equality holds in the second part of (3.1).*
If $f(0) + f^*(0) < \infty$, then equality holds in the second part of (3.1) if and only if $Q \perp P$.

Define the function $\tilde{f} : (0,\infty) \to \mathbb{R}$, $\tilde{f}(u) = \frac{1}{2}(f(u) + f^*(u))$. The following result is a refinement of the second inequality in Theorem 2 (see [3, Theorem 3]).

Theorem 3. *Let $f \in \mathcal{C}o([0,\infty))$ with $f(0) + f^*(0) < \infty$. Then*

$$0 \le I_f(Q,P) \le \tilde{f}(0)\, V(Q,P) \tag{3.2}$$

for any $Q, P \in \mathcal{P}$.

4. A General Divergence Measure

If $f : [0,\infty) \to \mathbb{R}$ is a general measurable function, then we may define the $f-$*divergence* in the same way, i.e., if $P, Q \in \mathcal{P}$, then

$$I_f(Q,P) = \int_X p(x)\, f\left[\frac{q(x)}{p(x)}\right] d\mu(x).$$

For a measurable function $g : [0,\infty) \to \mathbb{R}$, we may also define the $\delta-$*divergence* by the formula

$$\delta_g(Q,P) = \int_X [q(x) - p(x)]\, g\left[\frac{q(x)}{p(x)}\right] d\mu(x).$$

It is obvious that the $\delta-$divergence of a function g may be seen as the $f-$divergence of the function f, where $f(t) = (t-1)\, g(t)$ for $t \in [0,\infty)$.

If $f \in \mathcal{C}o([0,\infty))$ and since $f(t) = (t-1)\, g_\lambda(t)$, $t \in [0,\infty)$, we have

$$g_\lambda(t) := \begin{cases} \dfrac{f(t)}{t-1} & \text{if } \ t \in [0,1) \cup (1,\infty), \\ \\ \lambda & \text{if } \ t = 1; \end{cases} \tag{4.1}$$

and $\lambda \in [D_- f(1), D_+ f(1)]$, shows that for any $f \in \mathcal{C}o([0,\infty))$ we have

$$I_f(Q,P) = \delta g_\lambda(Q,P) \quad \text{for any} \quad P, Q \in \mathcal{P}, \tag{4.2}$$

i.e., *the $f-$divergence for any normalised continuous convex function $f : [0,\infty) \to \mathbb{R}$ may be seen as the $\delta-$divergence of the function g_λ defined by* (4.1).

In what follows, we point out some fundamental properties of the $\delta-$divergence.

Theorem 4. *Let $g : [0,\infty) \to \mathbb{R}$ be a measurable function on $[0,\infty)$ and $P, Q \in \mathcal{P}$. If there exists the constants m, M with*

$$-\infty < m \le g\left[\frac{q(x)}{p(x)}\right] \le M < \infty \tag{4.3}$$

for $\mu-$a.e. $x \in X$, then we have the inequality

$$|\delta_g(Q,P)| \le \frac{1}{2}(M - m)\, V(Q,P). \tag{4.4}$$

Proof. We observe that the following identity holds true

$$\delta_g(Q,P) = \int_X [q(x) - p(x)] \left[g\left[\frac{q(x)}{p(x)}\right] - \frac{m+M}{2} \right] d\mu(x) \tag{4.5}$$

By (4.3), we deduce that

$$\left|g\left[\frac{q(x)}{p(x)}\right]-\frac{m+M}{2}\right|\leq\frac{1}{2}(M-m)$$

for $\mu-$a.e. $x\in X$.

Taking the modulus in (4.5) we deduce

$$\begin{aligned}|\delta_g(Q,P)| &\leq \int_X |q(x)-p(x)|\left|g\left[\frac{q(x)}{p(x)}-\frac{m+M}{2}\right]\right|d\mu(x)\\ &\leq \frac{1}{2}(M-m)\int_X |q(x)-p(x)|\,d\mu(x)\\ &= \frac{1}{2}(M-m)V(Q,P)\end{aligned}$$

and the inequality (4.4) is proved. ∎

The following corollary is a natural consequence of the above theorem.

Corollary 1. *Let $g:[0,\infty)\to\mathbb{R}$ be a measurable function on $[0,\infty)$. If*

$$m:=ess\inf_{t\in[0,\infty)}g(t)>-\infty,\qquad M:=ess\sup_{t\in[0,\infty)}g(t)<\infty,$$

then for any $P,Q\in\mathcal{P}$, we have the inequality

$$|\delta_g(Q,P)|\leq\frac{1}{2}(M-m)V(Q,P). \tag{4.6}$$

Remark 2. *We know that, if $f:[0,\infty)\to\mathbb{R}$ is a normalised continuous convex function and if $\lim_{t\downarrow 0}f^*(t)=\lim_{u\downarrow 0}\left[uf\left(\frac{1}{u}\right)\right]=:f^*(0)$, then we have the inequality [Theorem 2.3]*

$$I_f(Q,P)\leq\frac{f(0)+f^*(0)}{2}V(Q,P), \tag{4.7}$$

for any $P,Q\in\mathcal{P}$. We can prove this inequality by the use of Corollary 1 as follows. We have

$$I_f(Q,P)=\delta g_\lambda(Q,P),$$

where

$$g_\lambda(t):=\begin{cases}\dfrac{f(t)}{t-1} & \text{if } t\in[0,1)\cup(1,\infty),\\ \\ \lambda & \text{if } t=1,\end{cases}$$

where $\lambda\in[D_-f(1),D_+f(1)]$ and $g_\lambda\in\mathcal{M}^{\ddagger}([0,\infty))$. We observe that for any $t\in[0,\infty)$, we have

$$g_\lambda(t)\geq\lim_{t\to 0+}g_\lambda(t)=-f(0)=m>-\infty$$

and

$$\begin{aligned}g_\lambda(t) &\leq \lim_{t\to+\infty}g_\lambda(t)=\lim_{t\to+\infty}\frac{f(t)}{t-1}=\lim_{u\to 0+}\left[\frac{f\left(\frac{1}{u}\right)}{\frac{1}{u}-1}\right]\\ &= \lim_{u\to 0+}\left[\frac{uf\left(\frac{1}{u}\right)}{1-u}\right]=f^*(0)=M<\infty.\end{aligned}$$

Applying Corollary 1 for $m=-f(0)$ and $M=f^*(0)$, we deduce the desired inequality (4.7).

The following result also holds.

Theorem 5. *Let $g:[0,\infty)\rightarrow\mathbb{R}$ be a measurable function on $[0,\infty)$ and $P,Q\in\mathcal{P}$. If there exists a constant K with $K>0$ such that*

$$\left|g\left(\frac{q(x)}{p(x)}\right)-g(1)\right|\leq K\left|\frac{q(x)}{p(x)}-1\right|^{\alpha}, \tag{4.8}$$

for $\mu-$a.e. $x\in X$, where $\alpha\in(0,\infty)$ is a given number, then we have the inequality

$$|\delta_g(Q,P)|\leq KI_{\chi^{\alpha+1}}(Q,P). \tag{4.9}$$

Proof. We observe that the following identity holds true

$$\delta_g(Q,P)=\int_X[q(x)-p(x)]\left[g\left[\frac{q(x)}{p(x)}\right]-g(1)\right]d\mu(x). \tag{4.10}$$

Taking the modulus in (4.10) and using the condition (4.8), we have successively

$$\begin{aligned}|\delta_g(Q,P)|&\leq\int_X|q(x)-p(x)|\left|g\left[\frac{q(x)}{p(x)}\right]-g(1)\right|d\mu(x)\\&\leq K\int_X[p(x)]^{-\alpha}|q(x)-p(x)|^{\alpha+1}d\mu(x)\\&\leq KI_{\chi^{\alpha+1}}(Q,P)\end{aligned}$$

and the inequality (4.9) is obtained. ∎

The following corollary holds.

Corollary 2. *Let $g:[0,\infty)\rightarrow\mathbb{R}$ be a measurable function on $[0,\infty)$ with the property that there exists a constant K with the property that*

$$|g(t)-g(1)|\leq K|t-1|^{\alpha}, \tag{4.11}$$

for a.e. $t\in[0,\infty)$, where $\alpha>0$ is a given number. Then for any $P,Q\in\mathcal{P}$, we have the inequality

$$|\delta_g(Q,P)|\leq KI_{\chi^{\alpha+1}}(Q,P). \tag{4.12}$$

Remark 3. *If the function $g:[0,\infty)\rightarrow\mathbb{R}$ is Hölder continuous with a constant $H>0$ and $\beta\in(0,1]$, i.e.,*

$$|g(t)-g(s)|\leq H|t-s|^{\beta},$$

for any $t,s\in[0,\infty)$, then obviously (4.7) holds with $K=H$ and $\alpha=\beta$.

If $g:[0,\infty)\rightarrow\mathbb{R}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|g(t)-g(s)|\leq L|t-s|,$$

for any $t,s\in[0,\infty)$, then

$$|\delta_g(Q,P)|\leq KI_{\chi^2}(Q,P), \tag{4.13}$$

for any $P,Q\in\mathcal{P}$.

Finally, if g is locally absolutely continuous and the derivative $g':[0,\infty)\rightarrow\mathbb{R}$ is essentially bounded, i.e., $\|g'\|_{[0,\infty),\infty}:=ess\sup_{t\in[0,\infty)}|g'(t)|<\infty$, then we have the inequality

$$|\delta_g(Q,P)|\leq\|g'\|_{[0,\infty),\infty}I_{\chi^2}(Q,P), \tag{4.14}$$

for any $P,Q\in\mathcal{P}$.

The following result concerning $f-$divergences for f convex functions holds.

Theorem 6. *Let $f : [0, \infty] \to \mathbb{R}$ be a continuous convex function on $[0, \infty)$. If $\lambda \in [D_- f(1), D_+ f(1)]$ ($\lambda = f'(1)$ if f is differentiable at $t = 1$), and there exists a constant $K > 0$ and $\alpha > 0$ such that*

$$|f(t) - \lambda(t-1)| \le K |t-1|^{\alpha+1}, \tag{4.15}$$

for any $t \in [0, \infty)$, then we have the inequality

$$0 \le I_f(Q, P) \le K I_{\chi^{\alpha+1}}(Q, P), \tag{4.16}$$

for any $P, Q \in \mathcal{P}$.

Proof. We have

$$I_f(Q, P) = \int_X [q(x) - p(x)] g_\lambda \left[\frac{p(x)}{q(x)}\right] d\mu(x) = \delta g_\lambda(Q, P),$$

where

$$g_\lambda(t) := \begin{cases} \dfrac{f(t)}{t-1} & \text{if } \ t \in [0,1) \cup (1, \infty), \\ \\ \lambda & \text{if } \ t = 1, \end{cases}$$

and $\lambda \in [D_- f(1), D_+ f(1)]$.

Applying Corollary 2 for g_λ, we deduce the desired result. ∎

5. The Positivity of $\delta-$Divergence for $g \in \mathcal{M}^{\ddagger}([0, \infty))$

The following result holds.

Theorem 7. *If $g \in \mathcal{M}^{\ddagger}([0, \infty))$, then $\delta_g(Q, P) \ge 0$ for any $P, Q \in \mathcal{P}$.*

Proof. We use the identity

$$\begin{aligned} \delta_g(Q,P) &= \int_X [q(x) - p(x)] g \left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \int_X p(x) \left[\frac{q(x)}{p(x)} - 1\right] g \left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \frac{1}{2} \int_X \int_X p(x) p(y) \left[\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right] \left[g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right] d\mu(x) \, d\mu(y). \end{aligned} \tag{5.1}$$

Since $g \in \mathcal{M}^{\ddagger}([0, \infty))$, then for any $t, s \in [0, \infty)$, we have

$$(t-s)(g(t) - g(s)) \ge 0$$

giving that

$$\left[\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right] \left[g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right] \ge 0$$

for any $x, y \in X$.

Using the representation (5.1), we deduce the desired result. ∎

The following corollary is a natural consequence of the above result.

Corollary 3. *If $f \in \mathcal{D}_0([0, \infty))$, then $I_f(Q, P) \ge 0$ for any $P, Q \in \mathcal{P}$.*

Proof. If $f \in \mathcal{D}_0([0,\infty))$, then there exists a $g \in \mathcal{M}^{\ddagger}([0,\infty))$ such that $f(t) = (t-1)g(t)$ for any $t \in [0,\infty)$. Then

$$\begin{aligned} I_f(Q,P) &= \int_X p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \int_X p(x)\left[\frac{q(x)}{p(x)} - 1\right] g\left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \delta_g(Q,P) \geq 0, \end{aligned}$$

and the proof is completed. ∎

In fact, the following improvement of Theorem 7 holds.

Theorem 8. *If* $g \in \mathcal{M}^{\ddagger}([0,\infty))$, *then*

$$\delta_g(Q,P) \geq \left|\delta_{|g|}(Q,P)\right| \geq 0, \tag{5.2}$$

for any $P, Q \in \mathcal{P}$.

Proof. Since g is monotonic nondecreasing, we have

$$\begin{aligned} &\left[\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right]\left[g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right] \\ &= \left|\left(\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right)\left(g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right)\right| \\ &\geq \left|\left(\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right)\left(\left|g\left[\frac{q(x)}{p(x)}\right]\right| - \left|g\left[\frac{q(y)}{p(y)}\right]\right|\right)\right| \end{aligned} \tag{5.3}$$

for any $x, y \in X$.

Multiplying (5.3) by $p(x)p(y) \geq 0$ and integrating on X^2, we deduce

$$\begin{aligned} &\int_X\int_X p(x)p(y)\left(\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right)\left[g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right] d\mu(x)\,d\mu(y) \\ &\geq \left|\int_X\int_X p(x)p(y)\left(\frac{q(x)}{p(x)} - \frac{q(y)}{p(y)}\right)\left(g\left[\frac{q(x)}{p(x)}\right] - g\left[\frac{q(y)}{p(y)}\right]\right) d\mu(x)\,d\mu(y)\right|. \end{aligned}$$

Using the representation (5.1) and the same identity for $|g|$, we deduce the desired inequality (5.2). ∎

Before we point out other possible refinements for the positivity inequality $\delta_g(Q,P) \geq 0$, where $g \in \mathcal{M}^{\ddagger}([0,\infty))$, we need the following divergence measure as well:

$$\bar{\delta}_h(Q,P) := \int_X |q(x) - p(x)|\, h\left[\frac{q(x)}{p(x)}\right] d\mu(x)$$

which will be called the *absolute* $\delta-$*divergence* generated by the function $h : [0,\infty) \to \mathbb{R}$ that is assumed to be measurable on $[0,\infty)$.

The following result holds.

Theorem 9. *If* $g \in \mathcal{M}^{\ddagger}([0,\infty))$, *then*

$$\begin{aligned} &\delta_g(Q,P) \\ &\geq \max\left\{\left|\bar{\delta}_g(Q,P) - V(Q,P)\,I_g(Q,P)\right|, \left|\bar{\delta}_{|g|}(Q,P) - V(Q,P)\,I_{|g|}(Q,P)\right|\right\} \geq 0, \end{aligned} \tag{5.4}$$

for any $P, Q \in \mathcal{P}$.

Proof. Since g is monotonic, we have

$$\begin{aligned}(5.5)\qquad & \left(\frac{q(x)}{p(x)}-\frac{q(y)}{p(y)}\right)\left(g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right)\\ &=\left|\left[\left(\frac{q(x)}{p(x)}-1\right)-\left(\frac{q(y)}{p(y)}-1\right)\right]\left[g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right]\right|\\ &\geq\begin{cases}\left|\left[\left|\frac{q(x)}{p(x)}-1\right|-\left|\frac{q(y)}{p(y)}-1\right|\right]\left[g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right]\right|\\ \\ \left|\left[\left|\frac{q(x)}{p(x)}-1\right|-\left|\frac{q(y)}{p(y)}-1\right|\right]\left[\left|g\left[\frac{q(x)}{p(x)}\right]\right|-\left|g\left[\frac{q(y)}{p(y)}\right]\right|\right]\right|\end{cases}\end{aligned}$$

for any $x,y\in X$.

If we multiply (5.5) by $p(x)p(y)\geq 0$ and integrate, we deduce

$$\begin{aligned}(5.6)\qquad & \int_X\int_X p(x)p(y)\left(\frac{q(x)}{p(x)}-\frac{q(y)}{p(y)}\right)\left(g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right)d\mu(x)\,d\mu(y)\\ &\geq\begin{cases}\Big|\int_X\int_X p(x)p(y)\left[\left|\frac{q(x)}{p(x)}-1\right|-\left|\frac{q(y)}{p(y)}-1\right|\right]\\ \qquad\times\left[g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right]d\mu(x)\,d\mu(y)\Big|\\ \\ \Big|\int_X\int_X p(x)p(y)\left[\left|\frac{q(x)}{p(x)}-1\right|-\left|\frac{q(y)}{p(y)}-1\right|\right]\\ \qquad\times\left[\left|g\left[\frac{q(x)}{p(x)}\right]\right|-\left|g\left[\frac{q(y)}{p(y)}\right]\right|\right]d\mu(x)\,d\mu(y)\Big|\end{cases}\end{aligned}$$

for any $x,y\in X$.

Now, observe that

$$\begin{aligned}&\int_X\int_X p(x)p(y)\left[\left|\frac{q(x)}{p(x)}-1\right|-\left|\frac{q(y)}{p(y)}-1\right|\right]\left[g\left[\frac{q(x)}{p(x)}\right]-g\left[\frac{q(y)}{p(y)}\right]\right]d\mu(x)\,d\mu(y)\\ &=\int_X\int_X p(x)p(y)\left[\left|\frac{q(x)}{p(x)}-1\right|g\left[\frac{q(x)}{p(x)}\right]+\left|\frac{q(y)}{p(y)}-1\right|g\left[\frac{q(y)}{p(y)}\right]\right]d\mu(x)\,d\mu(y)\\ &\quad-\int_X\int_X p(x)p(y)\left[\left|\frac{q(x)}{p(x)}-1\right|g\left[\frac{q(y)}{p(y)}\right]+\left|\frac{q(y)}{p(y)}-1\right|g\left[\frac{q(x)}{p(x)}\right]\right]d\mu(x)\,d\mu(y)\\ &=2\int_X p(y)\,d\mu(y)\int_X p(x)\left|\frac{q(x)}{p(x)}-1\right|g\left[\frac{q(x)}{p(x)}\right]d\mu(x)\\ &\quad-2\int_X p(x)\left|\frac{q(x)}{p(x)}-1\right|d\mu(x)\int_X p(y)\,g\left[\frac{q(y)}{p(y)}\right]d\mu(y)\\ &=2\left[\bar{\delta}_g(Q,P)-V(Q,P)\,I_g(Q,P)\right],\end{aligned}$$

and a similar identity holds for the quantity in the second branch of (5.6).

Finally, using the representation (5.1), we deduce the desired inequality (5.4). ∎

6. The Positivity of $\delta-$Divergence for $g\in\mathcal{L}e_1([0,\infty))$

The following result extending the positivity of $\delta-$divergence for monotonic functions, holds.

Theorem 10. *If $g\in\mathcal{L}e_1([0,\infty))$, then $\delta_g(Q,P)\geq 0$ for any $P,Q\in\mathcal{P}$.*

Proof. We use the identity

$$\delta_g(Q,P) = \int_X [q(x)-p(x)]\, g\left[\frac{q(x)}{p(x)}\right] d\mu(x) \tag{6.1}$$
$$= \int_X p(x)\left[\frac{q(x)}{p(x)}-1\right] g\left[\frac{q(x)}{p(x)}\right] d\mu(x)$$
$$= \int_X p(x)\left[\frac{q(x)}{p(x)}-1\right]\left[g\left[\frac{q(x)}{p(x)}\right]-g(1)\right] d\mu(x).$$

Since $g \in \mathcal{L}e_1([0,\infty))$, then for any $t \in [0,\infty)$ we have

$$(t-1)[g(t)-g(1)] \geq 0$$

giving that

$$\left(\frac{q(x)}{p(x)}-1\right)\left[g\left[\frac{q(x)}{p(x)}\right]-g(1)\right] \geq 0$$

for any $x \in X$.

Using the representation (6.1), we deduce the desired result. ∎

Corollary 4. *If $f \in \mathcal{O}_0([0,\infty))$, then $I_f(Q,P) \geq 0$ for any $P,Q \in \mathcal{P}$.*

Proof. If $f \in \mathcal{O}_0([0,\infty))$, then there exists a $g \in \mathcal{L}e_1([0,\infty))$ such that $f(t) = (t-1)\,g(t)$ for any $t \in [0,\infty)$. Then

$$\begin{aligned} I_f(Q,P) &= \int_X p(x)\, f\left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \int_X p(x)\left[\frac{q(x)}{p(x)}-1\right] g\left[\frac{q(x)}{p(x)}\right] d\mu(x) \\ &= \delta_g(Q,P) \geq 0, \end{aligned}$$

and the proof is completed. ∎

The following improvement of Theorem 10 holds.

Theorem 11. *If $g \in \mathcal{L}e_1([0,\infty))$, then*

$$\delta_g(Q,P) \geq \left|\delta_{|g|}(Q,P)\right| \geq 0 \tag{6.2}$$

for any $P,Q \in \mathcal{P}$.

Proof. Since $g \in \mathcal{L}e_1([0,\infty))$, we obviously have

$$\left[\frac{q(x)}{p(x)}-1\right]\left[g\left[\frac{q(x)}{p(x)}\right]-g(1)\right] \tag{6.3}$$
$$= \left|\left(\frac{q(x)}{p(x)}-1\right)\left(g\left[\frac{q(x)}{p(x)}\right]-g(1)\right)\right|$$
$$\geq \left|\left(\frac{q(x)}{p(x)}-1\right)\left(\left|g\left[\frac{q(x)}{p(x)}\right]\right|-|g(1)|\right)\right|.$$

Multiplying (6.3) by $p(x) \geq 0$ and integrating on X, we have

$$\begin{aligned} & \int_X p(x) \left[\frac{q(x)}{p(x)} - 1\right] \left[g\left[\frac{q(x)}{p(x)}\right] - g(1)\right] d\mu(x) \\ & = \int_X p(x) \left| \left(\frac{q(x)}{p(x)} - 1\right) \left(\left| g\left[\frac{q(x)}{p(x)}\right] \right| - |g(1)| \right) \right| d\mu(x) \\ & \geq \left| \int_X p(x) \left(\frac{q(x)}{p(x)} - 1\right) \left(\left| g\left[\frac{q(x)}{p(x)}\right] \right| - |g(1)| \right) d\mu(x) \right| \\ & = \left| \delta_{|g|}(Q,P) \right|, \end{aligned}$$

and the inequality (6.2) is proved. ∎

7. Bounds in Terms of the χ^2−Divergence

The following result may be stated.

Theorem 12. *Let $g : [0,\infty] \to \mathbb{R}$ be a differentiable function such that there exists the constants $\gamma, \Gamma \in \mathbb{R}$ with*

$$\gamma \leq g'(t) \leq \Gamma \quad \text{for any } \; t \in (0,\infty). \tag{7.1}$$

Then we have the inequality

$$\gamma D_{\chi^2}(Q,P) \leq \delta_g(Q,P) \leq \Gamma D_{\chi^2}(Q,P), \tag{7.2}$$

for any $P, Q \in \mathcal{P}$.

Proof. Consider the auxiliary function $h_\gamma : [0,\infty] \to \mathbb{R}$, $h_\gamma(t) := g(t) - \gamma(t-1)$. Obviously, h_γ is differentiable on $(0,\infty)$ and since, by (7.1),

$$h'_\gamma(t) = g'(t) - \gamma \geq 0$$

it follows that h_γ is monotonic nondecreasing on $[0,\infty)$.

Applying Theorem 7, we deduce

$$\delta_{h_\gamma}(Q,P) \geq 0 \quad \text{for any} \quad P, Q \in \mathcal{P}$$

and since

$$\begin{aligned} \delta_{h_\gamma}(Q,P) &= \delta_{g-\gamma(\cdot-1)}(Q,P) \\ &= \int_X [q(x) - p(x)] \left[g\left[\frac{q(x)}{p(x)}\right] - \gamma\left[\frac{q(x)}{p(x)} - 1\right]\right] d\mu(x) \\ &= \delta_g(Q,P) - \gamma D_{\chi^2}(Q,P), \end{aligned}$$

then the first inequality in (7.2) is proved.

The second inequality may be proven in a similar manner by using the auxiliary function $h_\Gamma : [0,\infty) \to \mathbb{R}$, $h_\Gamma(t) := \Gamma(t-1) - g(t)$. ∎

The following corollary is a natural application of the above theorem.

Corollary 5. *Let $f : [0,\infty] \to \mathbb{R}$ be a differentiable convex function on $(0,\infty)$ with $f(1) = 0$. If there exist the constants $\gamma, \Gamma \in \mathbb{R}$ with the property that:*

$$\gamma(t-1)^2 + f(t) \leq f'(t)(t-1) \leq f(t) + \Gamma(t-1)^2 \tag{7.3}$$

for any $t \in (0,\infty)$, then we have the inequality:

$$\gamma D_{\chi^2}(Q,P) \leq I_f(Q,P) \leq \Gamma D_{\chi^2}(Q,P) \tag{7.4}$$

for any $P, Q \in \mathcal{P}$.

Proof. We know that for any $P, Q \in \mathcal{P}$, we have (see for example (4.2)):

$$I_f(Q, P) = \delta_{g_{f'(1)}}(Q, P),$$

where

$$g_{f'(1)} = \begin{cases} \dfrac{f(t)}{t-1} & \text{if } t \in [0,1) \cup (1, \infty), \\ \\ f'(1) & \text{if } t = 1. \end{cases}$$

We observe that, by the hypothesis of the corollary, $g_{f'(1)}$ is differentiable on $(0, \infty)$ and

$$g'_{f'(1)}(t) = \frac{f'(t)(t-1) - f(t)}{(t-1)^2}$$

for any $t \in (0,1) \cup (1, \infty)$.

Using (7.3), we deduce that

$$\gamma \le g'_{f'(1)}(t) \le \Gamma$$

for $t \in (0, \infty)$, and applying Theorem 12 above, for $g = g_{f'(1)}$, we deduce the desired inequality (7.4). ∎

8. Bounds in Terms of the $J-$Divergence

We recall that the *Jeffreys divergence* (or $J-$divergence for short) is defined as

$$J(Q, P) := \int_X [q(x) - p(x)] \ln \left[\frac{q(x)}{p(x)}\right] d\mu(x), \tag{8.1}$$

where $P, Q \in \mathcal{P}$.

The following result holds.

Theorem 13. *Let $g : [0, \infty] \to \mathbb{R}$ be a differentiable function such that there exists the constants $\phi, \Phi \in \mathbb{R}$ with*

$$\phi \le t g'(t) \le \Phi \quad \textit{for any } \; t \in (0, \infty). \tag{8.2}$$

Then we have the inequality

$$\phi J(Q, P) \le \delta_g(Q, P) \le \Phi J(Q, P), \tag{8.3}$$

for any $P, Q \in \mathcal{P}$.

Proof. Consider the auxiliary function $h_\phi : [0, \infty) \to \mathbb{R}$, $h_\phi(t) := g(t) - \phi \ln t$. Obviously, h_ϕ is differentiable on $(0, \infty)$ and, by (8.2),

$$h'_\phi(t) = g'(t) - \frac{\phi}{t} = \frac{1}{t}[t g'(t) - \phi] \ge 0,$$

for any $t \in (0, \infty)$, showing that the function is monotonic nondecreasing on $(0, \infty)$.

Applying Theorem 7, we deduce

$$\delta_{h_\phi}(Q, P) \ge 0 \quad \text{for any } \; P, Q \in \mathcal{P}$$

and since

$$\begin{aligned} \delta_{h_\phi}(Q, P) &= \delta_{g - \phi \ln(\cdot)}(Q, P) \\ &= \int_X [q(x) - p(x)] \left[g\left[\frac{q(x)}{p(x)}\right] - \phi \ln \left[\frac{q(x)}{p(x)}\right] \right] d\mu(x) \\ &= \delta_g(Q, P) - \phi J(Q, P), \end{aligned}$$

then the first inequality in (8.3) is proved.

The second inequality may be proven in a similar manner by using the auxiliary function $h_\Phi : [0,\infty) \to \mathbb{R}$, $h_\Phi(t) := \Phi \ln t - g(t)$. ∎

The following corollary is a natural application of the above theorem.

Corollary 6. *Let $f : [0,\infty] \to \mathbb{R}$ be a differentiable convex function on $(0,\infty)$ with $f(1) = 0$. If there exist the constants $\phi, \Phi \in \mathbb{R}$ with the property that:*

$$\phi (t-1)^2 + tf(t) \leq t(t-1) f'(t) \leq tf(t) + \Phi (t-1)^2 \tag{8.4}$$

for any $t \in (0,\infty)$, then we have the inequality:

$$\phi J(Q,P) \leq I_f(Q,P) \leq \Phi J(Q,P) \tag{8.5}$$

for any $P, Q \in \mathcal{P}$.

The proof is similar to the one in Corollary 5 and we omit the details.

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